

Borromean-Rings Braiding Statistics and Topological Terms in Four-Dimensional Spacetime

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Braiding statistics data of topological excitations (e.g., anyons) play the role of “order parameters” of the long-range entanglement. In this Letter, we propose an exotic (3+1)-dimensional braiding process involving a particle and two loop-like excitations, which is coined “Borromean-Rings braiding”. In this process, a particle that carries $\prod_i \mathbb{Z}_{N_i}$ gauge charges moves around two unlinked loops that carry $\prod_i \mathbb{Z}_{N_i}$ gauge fluxes, such that its worldline and the two loops form the Borromean rings or more general Brunnian links. We show how the braiding phase depends on the Milnor’s invariant of Borromean-rings. We also propose a Topological Quantum Field Theory in (3+1)D where a topological term of “AAB” type is incorporated (A and B symbolically denote one-form and two-form $U(1)$ gauge fields, respectively). The “AAB” term monitors the Borromean-rings braiding phase and describes exotic topological order that goes beyond the Dijkgraaf-Witten cohomological construction. We further apply this braiding to characterize Symmetry Protected Topological phases protected by a *mixture* of usual (zero-form) and “generalized” (one-form) global symmetry.

Introduction.—In long-range entangled many-body ground states, conventional Landau-Ginzburg order parameters vanish. Instead, braiding statistics data of topological excitations can be used as key ingredients of “order parameters” [1]. These braiding statistics data are quantized and stable against any local perturbations, thereby being proposed to realize fault-tolerant topological quantum computation in (2+1)-dimensional (D) anyon systems [2–4]. In (3+1)D, particle-particle braiding is always trivial since the closed trajectory of a moving particle can be adiabatically shrunk to a point. However, the existence of loop excitations enriches possibilities of braiding statistics data. Physically, loop excitations can be simply modeled in the (3+1)D \mathbb{Z}_N gauge theory (in the deconfined Higgs phase [5, 6]), or, in more familiar terms, a type-II superconductor with a charge- N condensate coupled to a weakly fluctuating $U(1)$ gauge field [7]. The commonly known example of braiding statistics in (3+1)D is the particle-loop braiding statistics in which a particle (with charge $n \in \mathbb{Z}_N$) moves around a loop (with flux $2\pi m/N$, $m \in \mathbb{Z}_N$ due to the flux quantization in the above superconductor analogue). The process generates a Hopf link of the particle world-line and the loop. It is accompanied by the Aharonov-Bohm phase [8] $e^{i\frac{2\pi mn}{N}\mathcal{L}}$ ($\mathcal{L} \in \mathbb{Z}$ is the Hopf linking number [9]).

If one considers more complicated gauge group (i.e., $\prod_i \mathbb{Z}_{N_i}$), there exist exotic “multi-loop” braiding processes in which particles are not involved and loops are allowed to carry gauge fluxes of distinct \mathbb{Z}_{N_i} [10–24]. It has been recently discovered that these multi-loop braiding statistics data together with the particle-loop braiding statistics provide a complete characterization of $\prod_i \mathbb{Z}_{N_i}$ discrete gauge theories that are within the Dijkgraaf-Witten cohomological construction [25]. This progress has also significantly deepened our understanding on symmetry-protected topological phases (SPT), which are

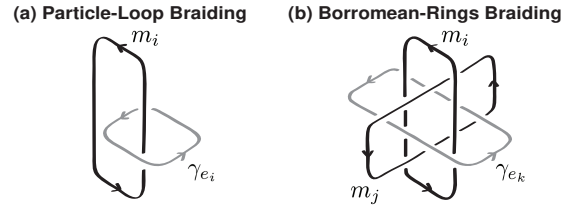


FIG. 1. Schematic representation of the particle-loop braiding (a) and Borromean-Rings (BR) braiding (b). In (b), a particle e_k with unit \mathbb{Z}_{N_k} gauge charge moves around two unlinked loops m_i and m_j that carry unit \mathbb{Z}_{N_i} and \mathbb{Z}_{N_j} gauge flux respectively. The particle trajectory (γ_{e_k}) and the two loops form the Borromean rings (or more general Brunnian links).

short-range entangled states with global symmetry [26–28]. One reason is that the global symmetry in SPTs can be “gauged” and the resulting braiding statistics data can be used to diagnose SPT orders [29].

In this Letter, we introduce a particle-loop-loop braiding process in (3+1)D that we coin “Borromean-Rings (BR) braiding”. In this process, a particle moves around two unlinked loops such that its world-line and the two loops form Borromean rings (Fig. 1b) or more general Brunnian links. Contrary to the particle-loop braiding where the world-line of the moving particle is always linked to the loop concerned, the world-line of the moving particle in the BR braiding is not linked to any of the two loops. Given a gauge group $\prod_i \mathbb{Z}_{N_i}$, we calculate the braiding phase associated with the BR braiding. The presence of BR braiding statistics data motivates us to introduce a new set of discrete gauge theories beyond the cohomological construction [25]. We propose a topological quantum field theory (TQFT) with the “ $A^i \wedge A^j \wedge B^k$ ” topological term, which monitors the BR braiding [$\{A^i\}$ and $\{B^i\}$ are 1-form and 2-form $U(1)$

gauge fields, respectively]. We provide a way to count the total number of beyond-cohomological discrete gauge theories. Finally, we propose SPTs protected by *mixed* $\prod_i \mathbb{Z}_{N_i}$ global symmetry where some \mathbb{Z}_{N_i} groups are the usual global symmetry (0-form) whereas the others are “generalized” global symmetry (1-form) [30]. After gauging, these SPTs are exactly turned to discrete gauge theories with the BR braiding. Therefore, the classification of such SPTs is obtained straightforwardly. Several future directions are summarized at the end of this Letter.

(3+1)D Borromean-Rings braiding.— Analogous to the \mathbb{Z}_N -valued braiding phase in the particle-loop braiding of discrete gauge theories, the BR braiding phase is quantized and periodic in an appropriate way. Below, on a general ground we attempt to extract several constraints on the braiding phase. A complete derivation will later be provided through the TQFT construction.

For simplicity, we assume that the BR braiding phase is given by a unit complex number $e^{i\Theta_{ij,k}}$ with the “braiding angle” $\Theta_{ij,k}$. In the braiding process, a particle (denoted by e_k) that carries unit gauge charge of \mathbb{Z}_{N_k} group moves around two unlinked loops m_i and m_j (carrying unit magnetic flux of \mathbb{Z}_{N_i} and \mathbb{Z}_{N_j} gauge groups respectively), which leads to the Borromean rings or more general (3-component) Brunnian links. 3-component Brunnian links are classified by Milnor’s invariant up to link homotopy [31, 32]: $\bar{\mu}(m_i, m_j, \gamma_{e_k})$ where γ_{e_k} denotes the world-line of e_k . Analogous to the particle-loop braiding, we assume the linear relation holds: $\Theta_{ij,k} = C_{ij,k} \bar{\mu}(m_i, m_j, \gamma_{e_k})$. This relation indicates that $C_{ij,k}$ can be interpreted as the braiding angle for the Borromean rings, i.e., the simplest example of Brunnian links with $\bar{\mu} = 1$. In other words, $C_{ij,k}$ provides universal properties of the underlying long-range entangled states that support the BR braiding. As a result, the classification of discrete gauge theories can be done by exhausting distinct $C_{ij,k}$ ’s.

Next, we show the anti-symmetric property of $\Theta_{ij,k}$. Consider a thought experiment in a BR braiding of e_k with m_i and m_j forming the Borromean rings with braiding angle $C_{ij,k}$. Imagine viewing the process from the opposite side, the positions of loops m_i and m_j are flipped, and so is the orientation of the three closed paths ($m_{i,j}$ and γ_{e_k}). It is equivalent to the BR braiding of e_k with m_j and m_i in opposite orientation with braiding angle $-C_{ji,k}$. Such equivalence implies that $C_{ji,k} = -C_{ij,k}$. In a general BR braiding with braiding angle $\Theta_{ij,k}$, if we interchange the positions of m_i and m_j ($\bar{\mu}(m'_i, m'_j, \gamma'_{e_k}) = \bar{\mu}(m_j, m_i, \gamma_{e_k})$), then braiding angle is $\Theta'_{ji,k} = C_{ji,k} \bar{\mu}(m'_j, m'_i, \gamma'_{e_k}) = C_{ji,k} \bar{\mu}(m'_i, m'_j, \gamma_{e_k}) = -\Theta_{ij,k}$. Hence interchanging the positions of the two loops flips the sign of the braiding angle.

We now show the quantization of the BR braiding phase. Consider carrying the particle e_k along its trajectory in an BR braiding by N_k times. By time translation, such braiding is the same as carrying $N_k e_k$ by once. Since

$\bar{\mu}(m_i, m_j, N_k \gamma_{e_k}) = N_k \bar{\mu}(m_i, m_j, \gamma_{e_k})$, the BR braiding angle is scaled by N_k . Similarly, if we wind m_i (m_j) along its locus by N_i (N_j) times, the braiding angle is multiplied by N_i (N_j). Note that N_i of m_i , N_j of m_j , N_k of e_k are all equivalent to vacuum. Since braiding is compatible to fusion, all the braiding angles must be trivial, that is, $N_i C_{ij,k} = 0 \bmod 2\pi$, $N_j C_{ij,k} = 0 \bmod 2\pi$, $N_k C_{ij,k} = 0 \bmod 2\pi$. Or equivalently, $C_{ij,k} = \frac{2\pi k_{ij,k}}{N_{ijk}}$, where $k_{ij,k}$ is an integer and N_{ijk} denotes the GCD (greatest common divisor) of N_i , N_j and N_k . As we have seen $C_{ji,k} = -C_{ij,k}$, $k_{ij,k}$ is anti-symmetric about interchanging i and j . Finally, we obtain the explicit form of the BR braiding phase:

$$\Theta_{ij,k} = \frac{2\pi k_{ij,k}}{N_{ijk}} \bar{\mu}(m_i, m_j, \gamma_{e_k}). \quad (1)$$

Topological terms of AAB type.— It is commonly believed that the low energy physics of long-range entangled states can be captured by certain types of TQFTs [2]. For example, the world-lines of two Abelian anyons in (2+1)D may form a Hopf link. The braiding phase associated to this process is described by Abelian Chern-Simons theory with the action $S = \frac{K^{ij}}{4\pi} \int A^i dA^j$ [1, 3, 33] where $K^{ij} = K^{ji} \in \mathbb{Z}$ and $\{A^i\}$ is a set of 1-form U(1) gauge fields [34]. Similarly, in (3+1)D, a loop excitation and the world-line of a moving particle may form a Hopf link. The braiding phase associated to this process can be described by the *BF* theory [35] with the action:

$$S_{\text{BF}} = \frac{N}{2\pi} \int B dA \quad (2)$$

which is the TQFT description of the topological limit of the \mathbb{Z}_N gauge theory [6]. B is a 2-form U(1) gauge field. The gauge transformations are: $A \rightarrow A + d\chi$ and $B \rightarrow B + dV$ where χ and V are 0-form and 1-form gauge parameters respectively. Integrating out either B or A imposes the local flatness condition as well as \mathbb{Z}_N -valued Wilson loop or Wilson surface: $\int_{S^1} A \in \frac{2\pi}{N} \mathbb{Z}$, $\int_{S^2} B \in \frac{2\pi}{N} \mathbb{Z}$ with S^1 and S^2 being a circle and a 2-sphere, respectively. In order to calculate the particle-loop braiding phase, one common way is to add the coupling term “ $-\int(AJ + B\sigma)$ ” and integrate out all gauge fields [36]. The resulting effective action is given by $\frac{2\pi}{N} \mathcal{L}$ in which the Hopf linking number $\mathcal{L} = \int d^{-1} J \sigma \in \mathbb{Z}$. Since the Feynman path integral evaluates the amplitude of the whole braiding process, the particle-loop braiding phase is given by $e^{i \frac{2\pi}{N} \mathcal{L}}$ that becomes trivial when $N = 1$.

In addition, multi-loop braiding statistics data [10–24] can also be nontrivial and constitute the *complete* description of the $\prod_i \mathbb{Z}_{N_i}$ gauge theories *within* the cohomological construction. The TQFT description involves topological terms of *AAdA* and *AAAA* types [13–22] in addition to the *BF* action (see Table I; the *AAAA* term only appears when G contains four or more \mathbb{Z}_{N_i} ’s).

Below, we move on to the physics of topological terms of *AAB* type. To this end, we should at least start

TABLE I. (3+1)D discrete gauge theories and braiding statistics ($G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$).

Braiding process & topological terms	$A^1 A^2 dA^1$	$A^2 A^3 dA^2$	$A^1 A^3 dA^1$	$A^1 A^2 dA^3$	$A^1 A^2 B^3$	$A^2 A^3 B^1$	$A^1 A^3 B^2$	# of discrete gauge theories
Multi-loop braiding	$\frac{A^1 A^2 dA^2}{(\mathbb{Z}_{N_{12}})^2}$	$\frac{A^2 A^3 dA^3}{(\mathbb{Z}_{N_{23}})^2}$	$\frac{A^1 A^3 dA^3}{(\mathbb{Z}_{N_{13}})^2}$	$\frac{A^2 A^3 dA^1}{(\mathbb{Z}_{N_{123}})^2}$	0	0	0	$(N_{12})^2 (N_{23})^2 (N_{13})^2 (N_{123})^2$
Borromean-Rings braiding	$\frac{A^1 A^2 dA^2}{(\mathbb{Z}_{N_{12}})^2}$	0	0	0	$\mathbb{Z}_{N_{123}}$	0	0	$(N_{123} - 1)(N_{12})^2$
	0	$(\mathbb{Z}_{N_{23}})^2$	0	0	0	$\mathbb{Z}_{N_{123}}$	0	$(N_{123} - 1)(N_{23})^2$
	0	0	$(\mathbb{Z}_{N_{13}})^2$	0	0	0	$\mathbb{Z}_{N_{123}}$	$(N_{123} - 1)(N_{13})^2$

with the gauge group $G = \prod_i \mathbb{Z}_{N_i}$ as the two A 's in AAB must come with different flavors. There are two choices: $A^1 A^2 B^1$ and $A^1 A^2 B^2$. For example, for the former, the total action is given by: $S = \sum_i \frac{N_i}{2\pi} \int B^i dA^i + \frac{c}{(2\pi)^2} \int A^1 A^2 B^1$, where c is a real number to be fixed. However, we argue that such theory is not legitimate. Actually, it is impossible to write down proper gauge transformations such that the theory is gauge-invariant while preserving the defining features of \mathbb{Z}_N gauge theory. To see this, one can integrate out either A^1 or B^1 , which fails to produce \mathbb{Z}_{N_1} -valued Wilson surface or loop. If we are to define the \mathbb{Z}_{N_1} structure, at least one of Wilson operators should be \mathbb{Z}_{N_1} -valued, which is fulfilled by previously studied $AAdA$ and $AAAA$ theories [15–17]. Therefore, we should *exclude* actions of the form $S = \sum_i \int \frac{N_i}{2\pi} B^i dA^i + S'$, where $\exists i$ such that S' part involves both B^i and A^i . Therefore, legitimate TQFTs with AAB terms do not exist for $G = \prod_i \mathbb{Z}_{N_i}$.

Let us move on to $G = \prod_i \mathbb{Z}_{N_i}$. Using the above *exclusion principle*, there are three classes of legitimate actions that involve AAB terms, as shown in the last three rows of Table I. One of the simplest actions is:

$$S_{\text{AAB}} = \sum_i \int \frac{N_i}{2\pi} B^i dA^i + \int \frac{c}{(2\pi)^2} A^1 A^2 B^3, \quad (3)$$

where c is to be fixed. The gauge transformations are:

$$A^i \rightarrow A^i + d\chi^i, \quad A^3 \rightarrow A^3 + d\chi^3 + \mathcal{Y}_3, \quad (4)$$

$$B^3 \rightarrow B^3 + dV^3, \quad B^i \rightarrow B^i + dV^i + \mathcal{Y}_i, \quad (5)$$

where $i = 1, 2$ only. $\chi^{1,2,3}$ and $V^{1,2,3}$ are 0-form and 1-form gauge parameters respectively. They satisfy the usual “winding number” conditions: $\int_{\mathbb{S}^1} d\chi^{1,2,3} \in 2\pi\mathbb{Z}$, $\int_{\mathbb{S}^2} dV^{1,2,3} \in 2\pi\mathbb{Z}$ [15]. $\mathcal{Y}_{1,2,3}$ are *nontrivially* induced by the $A^1 A^2 B^3$ term ($i, j = 1, 2$ only; see [37]):

$$\mathcal{Y}_i = \frac{c}{2\pi N_i} \epsilon^{ij} (A^j V^3 - \chi^j B^3 - \chi^j dV^3), \quad (6)$$

$$\mathcal{Y}_3 = -\frac{c}{2\pi N_3} (\epsilon^{ij} \chi^i A^j + \chi^1 d\chi^2). \quad (7)$$

Apparently, once $c = 0$, $\mathcal{Y}_{1,2,3}$ constantly vanish, reproducing the gauge transformations of the BF action (2).

Integrating out $B^{1,2}$ and A^3 leads to quantized Wilson loop/surface ($i = 1, 2$): $\int_{\mathbb{S}^1} A^i \in \frac{2\pi}{N_i} \mathbb{Z}$, $\int_{\mathbb{S}^2} B^3 \in$

$\frac{2\pi}{N_3} \mathbb{Z}$, which enforces the quantization of the $A^1 A^2 B^3$ term in Eq. (3) on a spacetime manifold $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^2$: $\int \frac{c}{(2\pi)^2} A^1 A^2 B^3 = \frac{c}{(2\pi)^2} \int_{\mathbb{S}^1} A^1 \int_{\mathbb{S}^1} A^2 \int_{\mathbb{S}^2} B^3 \in \frac{2\pi c}{N_1 N_2 N_3} \mathbb{Z}$. Since the partition function should be unaltered under $S_{\text{AAB}} \rightarrow S_{\text{AAB}} + 2\pi$, this quantization indicates that there is a periodic redundancy for c : $c \sim c + N_1 N_2 N_3$. On the other hand, under the gauge transformations of $A^{1,2}$ and B^3 in Eqs. (4) and (5), the action must be invariant modulo 2π . By imposing the “winding number” conditions, we find that c is divisible by $N_2 N_3$, $N_1 N_3$ and $N_1 N_2$, i.e.,

$$c = \frac{N_1 N_2 N_3}{N_{123}} t, \quad t \in \mathbb{Z}_{N_{123}} \quad (8)$$

which means that there are totally N_{123} distinct values for c as the coefficient of the $A^1 A^2 B^3$ term (Table I).

Comparing with Eq. (1), observe that $\frac{c}{N_1 N_2 N_3}$ in Eq. (8) is equal to $C^{12,3}$. Below we will make a connection between the AAB terms and the Milnor’s invariant [31] of the Borromean Rings (or more general Brunnian links). Analogous to the particle-loop braiding of Eq. (2), we introduce currents $J^{1,2,3}$ for particles $e_{1,2,3}$ and currents $\sigma^{1,2,3}$ for loops $m_{1,2,3}$ (see Fig. 1) by adding a coupling term S_{cpl} in Eq. (3):

$$S_{\text{cpl}} = - \int \left[\sum_i \left(A^i J^i + \mathcal{B}^i \sigma^i \right) + \mathcal{A}^3 J^3 + B^3 \sigma^3 \right], \quad (9)$$

where $\mathcal{A}^3 \equiv A^3 - \frac{c}{4\pi N_3} A^i d^{-1} A^j \epsilon^{ij}$ and $\mathcal{B}^i \equiv B^i - \frac{c}{4\pi N_i} \epsilon^{ij} (A^j d^{-1} B^3 - B^3 d^{-1} A^j)$ with $i, j = 1, 2$. Under gauge transformations, these “modified” gauge potentials are changed by total derivatives: $\mathcal{A}^3 \rightarrow \mathcal{A}^3 - \frac{c}{4\pi N_3} d(\chi^i d^{-1} A^j \epsilon^{ij} + \chi^1 \chi^2)$ and $\mathcal{B}^i \rightarrow \mathcal{B}^i - \frac{\epsilon^{ij} c}{4\pi N_i} d(\chi^j d^{-1} B^3 - V^3 d^{-1} A^j + \chi^j V^3)$. One can verify that S_{cpl} is gauge invariant as long as all currents are conserved, namely $dJ^{1,2,3} = 0$ and $d\sigma^{1,2,3} = 0$. Before proceeding further, it is meaningful to observe that both $\mathcal{B}^i \sigma^i$ and $\mathcal{A}^3 J^3$ are “non-minimal” gauge couplings that are different from the familiar minimal coupling terms, e.g., $A^i J^i$ and $B^3 \sigma^3$. Integrating out all gauge fields in $S_{\text{AAB}} + S_{\text{cpl}}$ (first $B^{1,2}$, A^3 and then $A^{1,2}$, B^3), we obtain the effective action that is physically identified as the total braiding angle Θ_{Braid} [1]: $\Theta_{\text{Braid}} = \mathcal{S}_{\text{hopf}} + \mathcal{S}_{\text{br}}$. Here, $\mathcal{S}_{\text{hopf}} = \sum_i \frac{2\pi}{N_i} \mathcal{L}^i$ provides the Aharonov-Bohm phase $e^{i\mathcal{S}_{\text{hopf}}}$ of the particle-loop braiding. The Hopf linking

number $\mathcal{L}^i = \int d^{-1} J^i \sigma^i$. $\mathcal{S}_{\text{br}} = \frac{2\pi t}{N_{123}} \bar{\mu}$ describes the non-trivial BR braiding of $\sigma^{1,2}$ and J^3 with the braiding phase $e^{i\mathcal{S}_{\text{br}}}$, consistent to Eq. (1). Here, the integral formula for $\bar{\mu}$ is given by $\int d^{-1} \sigma^1 d^{-1} \sigma^2 d^{-1} J^3 - \int \frac{1}{2} (d^{-1} \sigma^1 d^{-2} \sigma^2 - d^{-1} \sigma^2 d^{-2} \sigma^1) J^3 - \int \frac{1}{2} \sigma^1 (d^{-1} \sigma^2 d^{-2} J^3 - d^{-1} J^3 d^{-2} \sigma^2) - \int \frac{1}{2} \sigma^2 (d^{-1} J^3 d^{-2} \sigma^1 - d^{-1} \sigma^1 d^{-2} J^3)$ which is geometrically identified as the Milnor's invariant [37].

Next, we ask how many topological discrete gauge theories there are, which are characterized by nontrivial BR braiding (Table I). For $G = \prod_i \mathbb{Z}_{N_i}$, there are totally N_{123} distinct $A^1 A^2 B^3$ terms that support BR braiding of m_1 , m_2 and e_3 . Excluding the trivial case $t = 0$ in Eq. (8), there are $(N_{123} - 1)$ non-trivial BR braiding phases. For each of them, we can equip $(N_{12})^2$ theories arising from $A^1 A^2 dA^1$ and $A^1 A^2 dA^2$ that support multi-loop braidings of m_1 and m_2 . Similarly, we can introduce $A^1 A^3 B^2$ for m_3 , m_1 and e_2 and $A^2 A^3 B^1$ for m_2 , m_3 and e_1 . However, each pair of AAB terms, e.g., $A^1 A^2 B^3$ and $A^2 A^3 B^1$, is disallowed to appear due to the exclusion principle. As such, for $G = \prod_i \mathbb{Z}_{N_i}$, the total number of gauge theories supporting nontrivial BR braiding is obtained by *summing* over the above three individual contributions: $(N_{123}-1) \sum_{i < j}^3 (N_{ij})^2$ in Table I. A more intricate case with $G = \prod_i \mathbb{Z}_{N_i}$ is shown in [37].

SPTs with mixed global symmetry.— The above discussion on beyond-cohomology discrete gauge theories with G also implies existence of exotic SPTs beyond the group cohomological classification [26] due to the gauging idea [29]. To illustrate the gauging process, we start with an “almost trivial” field theory:

$$S_{\text{aab}} = \sum_i^3 \int \frac{1}{2\pi} b^i da^i + \int \frac{c}{(2\pi)^2} a^1 a^2 b^3, \quad (10)$$

where $b^{1,2,3}$ and $a^{1,2,3}$ are 2-form and 1-form U(1) gauge fields respectively. At first sight, the structure of this action differs from Eq. (3) only by the coefficients of the BF terms. Thus, both particle-loop braiding and BR braiding of Eq. (10) are trivial, as shown in the expression of Θ_{Braid} where both terms are 0 mod 2π . Indeed, due to Eq. (8), we have $c \in \mathbb{Z}$ and $c \sim c + 1$, indicating that any integral c is equivalent to $c = 0$ in a sense of trivial braiding statistics data. Despite that, Eq. (10) is a good starting point for field theoretic descriptions of SPTs [15, 38–43] since the latter, by definition, do not support nontrivial braiding statistics. Next, we incorporate global symmetry $G = \prod_i \mathbb{Z}_{N_i}$ by considering coupling to external gauge fields described by:

$$S_{\text{sym}} = - \sum_i^2 \int \frac{1}{2\pi} A^i db^i - \int \frac{1}{2\pi} B^3 da^3. \quad (11)$$

Here, the 1-form U(1) gauge fields $A^{1,2}$ are the usual probe fields of $\mathbb{Z}_{N_{1,2}}$ and are constrained by $\int_{\mathbb{S}^1} A^{1,2} \in \frac{2\pi}{N_{1,2}} \mathbb{Z}$. The symmetry charges are carried by currents of

particles: $\frac{1}{2\pi} db^{1,2}$. Such coupling terms have been studied thoroughly [15–17, 44]. However, the 2-form U(1) gauge field B^3 is an unusual probe field of \mathbb{Z}_{N_3} and is constrained by $\int_{\mathbb{S}^2} B^3 \in \frac{2\pi}{N_3} \mathbb{Z}$. The symmetry charges are carried by loops $\frac{1}{2\pi} da^3$ — spatially extended matter fields. Since a p -form symmetry acts on charge of dimension p [30, 45, 46], the charge of a 0-form symmetry (e.g., $\mathbb{Z}_{N_{1,2}}$) is carried by a point object, whereas the charge of a 1-form symmetry (e.g., \mathbb{Z}_{N_3}) is carried by a line object. We can thus conclude that SPTs described by $S_{\text{aab}} + S_{\text{sym}}$ support *mixed* global symmetry. Several supplemental discussions are provided in [37], including the gauge transformations and a condensation scenario [15, 42, 47, 48] where condensations of bosons [15] and loops [42] nontrivially coexist.

In order to diagnose SPT orders, one may further integrate out a^i, b^i , which exactly leads to Eq. (3). This process can be summarized as follows:

$$S_{\text{aab}} + S_{\text{sym}} \xrightleftharpoons[\text{Un-gauging}]{\text{Gauging}} S_{\text{AAB}}, \quad (12)$$

where ungauging denotes the reverse process. It should be noted that $B^{1,2}$ and A^3 are introduced as Lagrange's multipliers in order to enforce the constraints on $A^{1,2}, B^3$. It is known that c in the gauged theory is classified by Eq. (8), which enlarges the period (from 1 to $N_1 N_2 N_3$). As a result, we can have N_{123} different SPT phases labeled by $t \in \mathbb{Z}_{N_{123}}$. Any two of them, despite the same trivial braiding statistics in the bulk, differ from each other by braiding statistics *after gauging* G [29]. To sum up, t can be used to label nontrivial SPTs protected by mixed global symmetries (denoted by SPT*).

Without loss of generality, suppose: $G = G_0 \times G_1$ with 0-form symmetry $G_0 = \prod_{i \in U} \mathbb{Z}_{N_i}$ and 1-form symmetry $G_1 = \prod_{r \in V} \mathbb{Z}_{N_r}$. $U = \{1, \dots, p\}$ and $V = \{p+1, \dots, p+q\}$ with $p, q \in \mathbb{Z}, p \geq 2$. In order to satisfy the exclusion principle, we may first consider aab terms of the form $a^i a^j b^r$ with a *fixed* $r \in V$. For example, if $r = p+1$, then, all aab^{p+1} terms are allowed to simultaneously appear in the action, which gives $\prod_{i < j \in U} N_{ijp+1}$ SPTs. But one of them corresponds to the action where all coefficients of aab terms vanish. Hence, this trivial action should be removed, leading to $(\prod_{i < j \in U} N_{ijp+1} - 1)$ SPT*s. In addition, we can freely add $aada$ and $aaaa$ terms in the action as long as the indices of a 's are within the set U . Therefore, for $r = p+1$, there are $(\prod_{i < j \in U} N_{ijp+1} - 1) |H^4(G_0, U(1))|$ SPT*s, where $|H^4(G_0, U(1))|$ is the order of the cohomology group H^4 [26] that counts the total number of topologically distinct $aada$ and $aaaa$ terms [15]. Since aab^r terms of different r 's are allowed to appear simultaneously, we obtain the total number of SPT*s:

$$|H^4(G_0, U(1))| \times \prod_{r \in V} \left(\prod_{i < j \in U} N_{ijr} - 1 \right). \quad (13)$$

We remark that, in order to obtain more beyond-cohomology theories or even fermionic theories, topological terms of bb [30, 35, 42, 49–51] and $dada$ [52–63] types should also be considered.

Outlook.— One future direction is to study boundary anomaly for the AAB term and also explore vast types of mixed terms in higher dimensions, e.g., $AAAB$, $AdAB$ and AAC (C denotes 3-form gauge fields) in (4+1)D [64]. A proper dimensional reduction will be useful, e.g., comparing the Wilson algebra of AAB (beyond-cohomology) and that of AAA [65] (within-cohomology). For SPTs with mixed global symmetry, it is meaningful to investigate non-linear σ model descriptions [66]. While the context here is the study of topological phases of matter in condensed matter physics, these topological terms and mixed global symmetry can also be applied to fluid dynamics and dense plasma, e.g., [67, 68].

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Supplemental Material

In the following, we provide several useful supplemental information. $N_{ijk\dots}$ is the greatest common divisor of N_i, N_j, N_k, \dots

1. Gauge transformations of the TQFT “ $BdA + AAB$ ”;
2. Geometric interpretation of the \mathcal{S}_{br} ;
3. Counting discrete gauge theories with nontrivial BR braiding: A concrete example for $G = \prod_i^4 \mathbb{Z}_{N_i}$;
4. Gauge transformations in the “almost trivial” field theory “ $bda + aab$ ” of SPTs;
5. Coexistence of loop condensations and boson condensations.

Section-1 Gauge transformations of the TQFT “ $BdA + AAB$ ”

tation):

In the main text, the gauge transformations in the action Eq. (3) are given by: $A^i \rightarrow A^i + d\chi^i$, $A^3 \rightarrow A^3 + d\chi^3 + \mathcal{Y}_3$, $B^3 \rightarrow B^3 + dV^3$, $B^i \rightarrow B^i + dV^i + \mathcal{Y}_i$. $\chi^{1,2,3}$ are 0-form (scalar); $A^{1,2,3}, V^{1,2,3}, \mathcal{Y}_3$ are 1-form (vector); $B^{1,2,3}, \mathcal{Y}_{1,2}$ are 2-form (tensor).

The expressions of $\mathcal{Y}_{1,2,3}$ are ($i = 1, 2$ only) are provided in the main text. Below, we sketch several key steps towards these expressions. First, under the gauge transformations, the action is changed to (the wedge product operator “ \wedge ” is omitted for simplifying notation only):

$$\begin{aligned}
 S_{\text{AAB}} \rightarrow S_{\text{AAB}} = & \int \frac{N_1}{2\pi} (B^1 + dV^1 + \mathcal{Y}_1) d(A^1 + d\chi^1) \\
 & + \int \frac{N_2}{2\pi} (B^2 + dV^2 + \mathcal{Y}_2) d(A^1 + d\chi^2) \\
 & + \int \frac{N_3}{2\pi} (B^3 + dV^3) d(A^3 + d\chi^3 + \mathcal{Y}_3) \\
 & + \int \frac{c}{4\pi^2} (A^1 + d\chi^1)(A^2 + d\chi^2)(B^3 + dV^3).
 \end{aligned} \tag{14}$$

Expanding the above expression, we can find the change of S_{AAB} (the integral “ \int ” is omitted for simplifying notation):

$$\begin{aligned}
 \Delta S_{\text{AAB}} = & \frac{N_1}{2\pi} \mathcal{Y}_1 dA^1 + \frac{N_2}{2\pi} \mathcal{Y}_2 dA^2 + \frac{N_3}{2\pi} B^3 d\mathcal{Y}_3 + \frac{c}{4\pi^2} A^1 A^2 dV^3 \\
 & + \frac{c}{4\pi^2} A^1 d\chi^2 B^3 + \frac{c}{4\pi^2} A^1 d\chi^3 dV^3 + \frac{c}{4\pi^2} d\chi^1 A^2 B^3 \\
 & + \frac{c}{4\pi^2} d\chi^1 A^2 dV^3 + \frac{c}{4\pi^2} d\chi^1 d\chi^2 B^3,
 \end{aligned} \tag{15}$$

where all total derivative terms are neglected. Next, we can reorganize all terms as $\Delta S_{\text{AAB}} = \mathcal{C}_1 dA^1 + \mathcal{C}_2 dA^2 + \mathcal{C}_3 dB^3$ where $\mathcal{C}_{1,2,3}$ are coefficients. In order to maintain gauge invariance, all coefficients should vanish: $\mathcal{C}_{1,2,3} = 0$. For this purpose, we may deform each term appropriately by following the standard rules of differential forms: (i) integration-by-parts is repeatedly used ($d(\mathbf{f}_\ell \wedge \mathbf{f}_{\ell'}) = d\mathbf{f}_\ell \wedge \mathbf{f}_{\ell'} + (-1)^\ell \mathbf{f}_\ell \wedge d\mathbf{f}_{\ell'}$) and all total derivatives of 4-form are neglected; (ii) $d^2 = 0$ is applied; (iii) the sign convention is given by $\mathbf{f}_\ell \wedge \mathbf{f}_{\ell'} = (-1)^{\ell\ell'} \mathbf{f}_{\ell'} \wedge \mathbf{f}_\ell$ where \mathbf{f}_ℓ is an ℓ -form variable and $\mathbf{f}_{\ell'}$ is an ℓ' -form variable. The results are given by (numerical prefactors are

not written explicitly):

$$B^3 d\mathcal{Y}_3 = -dB^3 \mathcal{Y}_3 = \mathcal{Y}_3 dB^3, \quad (16)$$

$$\begin{aligned} A^1 A^2 dV^3 &= -d(A^1 A^2) V^3 = -dA^1 A^2 V^3 + A^1 dA^2 V^3 \\ &= -A^2 V^3 dA^1 + A^1 V^3 dA^2, \end{aligned} \quad (17)$$

$$\begin{aligned} A^1 d\chi^2 B^3 &= -d\chi^2 A^1 B^3 = \chi^2 d(A^1 B^3) \\ &= \chi^2 dA^1 B^3 - \chi^2 A^1 dB^3 \\ &= \chi^2 B^3 dA^1 - \chi^2 A^1 dB^3, \end{aligned} \quad (18)$$

$$\begin{aligned} A^1 d\chi^2 dV^3 &= A^1 d(\chi^2 dV^3) = dA^1 \chi^2 dV^3 \\ &= \chi^2 dV^3 dA^1, \end{aligned} \quad (19)$$

$$\begin{aligned} d\chi^1 A^2 B^3 &= -\chi^1 d(A^2 B^3) = -\chi^1 dA^2 B^3 + \chi^1 A^2 dB^3 \\ &= -\chi^1 B^3 dA^2 + \chi^1 A^2 dB^3, \end{aligned} \quad (20)$$

$$\begin{aligned} d\chi^1 A^2 dV^3 &= d\chi^1 dV^3 A^2 = d(\chi^1 dV^3) A^2 \\ &= -\chi^1 dV^3 dA^2, \end{aligned} \quad (21)$$

$$d\chi^1 d\chi^2 B^3 = d(\chi^1 d\chi^2) B^3 = \chi^1 d\chi^2 dB^3. \quad (22)$$

Then, we obtain $\mathcal{C}_{1,2,3}$:

$$\mathcal{C}_1 = \frac{N_1}{2\pi} \mathcal{Y}_1 - \frac{c}{4\pi^2} A^2 V^3 + \frac{c}{4\pi^2} \chi^2 B^3 + \frac{c}{4\pi^2} \chi^2 dV^3 = 0, \quad (23)$$

$$\mathcal{C}_2 = \frac{N_2}{2\pi} \mathcal{Y}_2 + \frac{c}{4\pi^2} A^1 V^3 - \frac{c}{4\pi^2} \chi^1 B^3 - \frac{c}{4\pi^2} \chi^1 dV^3 = 0, \quad (24)$$

$$\mathcal{C}_3 = \frac{N_3}{2\pi} \mathcal{Y}_3 + \frac{c}{4\pi^2} \chi^1 A^2 - \frac{c}{4\pi^2} \chi^2 A^1 + \frac{c}{4\pi^2} \chi^1 d\chi^2 = 0 \quad (25)$$

which directly lead to the expressions of $\mathcal{Y}_{1,2,3}$ in the main text.

Section-2: Geometric interpretation of \mathcal{S}_{br}

In this supplemental material, we discuss the geometric interpretation of the \mathcal{S}_{br} in the effective action. If the position of the two loops are fixed, \mathcal{S}_{br} reduces to the three dimensional spatial integral:

$$\begin{aligned} \mathcal{S}_{\text{br}} &= \frac{2\pi t}{N_{123}} \int \left[d^{-1} \sigma^1 d^{-1} \sigma^2 d^{-1} j^3 \right] \\ &\quad - \left[\frac{1}{2} j^3 (d^{-1} \sigma^1 d^{-2} \sigma^2 - d^{-1} \sigma^2 d^{-2} \sigma^1) \right] \\ &\quad - \left[\frac{1}{2} \sigma^1 (d^{-1} \sigma^2 d^{-2} j^3 - d^{-1} j^3 d^{-2} \sigma^2) \right] \\ &\quad - \left[\frac{1}{2} \sigma^2 (d^{-1} j^3 d^{-2} \sigma^1 - d^{-1} \sigma^1 d^{-2} j^3) \right] \end{aligned} \quad (26)$$

where j^3 describes the trajectory of the particle e_3 in three dimension space. We denote m_1, m_2, γ_{e_3} as $\gamma^1, \gamma^2, \gamma^3$ here. The integrals in the four square brackets have different geometric meaning. Let $\mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3$ be the

Seifert surfaces bound by $\gamma^1, \gamma^2, \gamma^3$. For the first square bracket, the integral is

$$\begin{aligned} \int \left[d^{-1} \sigma^1 d^{-1} \sigma^2 d^{-1} j^3 \right] &= \int \delta(\mathbf{S}^1) \delta(\mathbf{S}^2) \delta(\mathbf{S}^3) \\ &= \int \delta(\mathbf{S}^1 \cap \mathbf{S}^2 \cap \mathbf{S}^3) \\ &= \text{Int} \# (\mathbf{S}^1 \cap \mathbf{S}^2 \cap \mathbf{S}^3) \end{aligned} \quad (27)$$

which is simply the sum of signed intersection of $\mathbf{S}^1 \cap \mathbf{S}^2 \cap \mathbf{S}^3$ with each sign determined by the orientation of the normals of $\mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3$ at the intersections. For the square square bracket, the integral is

$$\begin{aligned} &\frac{1}{2} \int j^3 (d^{-1} \sigma^1 d^{-2} \sigma^2 - d^{-1} \sigma^2 d^{-2} \sigma^1) \\ &= \frac{1}{2} \int \delta(\gamma^3) (\delta(\mathbf{S}^1) d^{-2} \sigma^2 - \delta(\mathbf{S}^2) d^{-2} \sigma^1) \\ &= \frac{1}{2} \int \delta(\gamma^3) \delta(\mathbf{S}^1) \int_{\gamma^3[x_o, x]} \delta(\mathbf{S}^2) \\ &\quad + \frac{1}{2} \int \delta(\bar{\gamma}^3) \delta(\mathbf{S}^2) \int_{\bar{\gamma}^3[x_o, x]} \delta(\mathbf{S}^1) \\ &= \frac{1}{2} \int \delta(\gamma^3 \cap \mathbf{S}^1) \text{Int} \# (\gamma^3[x_o, x] \cap \mathbf{S}^2) \\ &\quad + \frac{1}{2} \int \delta(\bar{\gamma}^3 \cap \mathbf{S}^2) \text{Int} \# (\bar{\gamma}^3[x_o, x] \cap \mathbf{S}^1), \end{aligned} \quad (28)$$

where $\bar{\gamma}^3$ is the loop formed by reversing the loop γ^3 , x_o is a reference point on the loop γ^3 and x is another point on the loop γ^3 , and $[x_o, x] \gamma^3$ is a segment of γ^3 starting from x_o and ending at x . The two integrals both give the number of occurrence of \mathbf{S}^1 after \mathbf{S}^2 walking along γ^3 and is denoted as $\text{Occ} \# (\mathbf{S}^1, \mathbf{S}^2; \gamma^3)$. Hence we can combine the two integrals to obtain the equation

$$\begin{aligned} &\frac{1}{2} \int j^3 (d^{-1} \sigma^1 d^{-2} \sigma^2 - d^{-1} \sigma^2 d^{-2} \sigma^1) \\ &= \int \delta(\gamma^3 \cap \mathbf{S}^1) \text{Int} \# (\gamma^3[x_o, x] \cap \mathbf{S}^2) \\ &= \text{Occ} \# (\mathbf{S}^1, \mathbf{S}^2; \gamma^3) \end{aligned} \quad (29)$$

For the third and fourth square brackets, the integral can be interpreted similarly as the integral in the second bracket. Finally, we get the geometric interpretation

$$\begin{aligned} \mathcal{S}_{\text{br}} &= \text{Int} \# (\mathbf{S}^1 \cap \mathbf{S}^2 \cap \mathbf{S}^3) \\ &\quad - \left(\text{Occ} \# (\mathbf{S}^1, \mathbf{S}^2; \gamma^3) + \text{Occ} \# (\mathbf{S}^2, \mathbf{S}^3; \gamma^1) \right. \\ &\quad \left. + \text{Occ} \# (\mathbf{S}^3, \mathbf{S}^1; \gamma^2) \right) \end{aligned} \quad (30)$$

which is simply the Milnor's triple linking number $\bar{\mu}(\gamma^1, \gamma^2, \gamma^3)$.

Section-3 Counting discrete gauge theories with nontrivial BR braiding: a concrete example for
 $G = \prod_i^4 \mathbb{Z}_{N_i}$

When $G = \prod_i^4 \mathbb{Z}_{N_i}$, we need to consider totally 12 AAB terms that can be categorized into four classes:

1. $A^2A^3B^1, A^2A^4B^1, A^3A^4B^1$;
2. $A^1A^3B^2, A^1A^4B^2, A^3A^4B^2$;
3. $A^1A^2B^3, A^1A^4B^3, A^2A^4B^3$;
4. $A^1A^2B^4, A^1A^3B^4, A^2A^3B^4$.

Let us first consider the action where only one nonzero AAB term is involved. In this case, we have:

1. for $A^2A^3B^1$: $(N_{231}-1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2$;
2. for $A^2A^4B^1$: $(N_{241}-1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2$;
3. for $A^3A^4B^1$: $(N_{341}-1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2$;
4. for $A^1A^3B^2$: $(N_{132}-1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2$;
5. for $A^1A^4B^2$: $(N_{142}-1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2$;
6. for $A^3A^4B^2$: $(N_{342}-1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2$;
7. for $A^1A^2B^3$: $(N_{123}-1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2$;
8. for $A^1A^4B^3$: $(N_{143}-1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2$;
9. for $A^2A^4B^3$: $(N_{243}-1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2$;
10. for $A^1A^2B^4$: $(N_{124}-1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2$;
11. for $A^1A^3B^4$: $(N_{134}-1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2$;
12. for $A^2A^3B^4$: $(N_{234}-1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2$.

For example, for $A^2A^3B^1$, the coefficient of this term has $N_{231} - 1$ nonzero choices. For each choice, one may attach $AAdA$ terms that do not contain A^1 , i.e., $A^2A^3dA^2$, $A^2A^3dA^3$, $A^2A^4dA^2$, $A^2A^4dA^4$, $A^3A^4dA^3$, $A^3A^4dA^4$, $A^2A^3dA^4$ and $A^2A^4dA^3$. Note that $A^3A^4dA^2$ is not linearly independent on $A^2A^3dA^4$ and $A^2A^4dA^3$.

Then, if there are two nonzero AAB terms in the action, we have:

1. for $A^1A^2B^3 + A^1A^2B^4$: $(N_{123}-1)(N_{124}-1)(N_{12})^2$;
2. for $A^1A^3B^2 + A^1A^3B^4$: $(N_{132}-1)(N_{134}-1)(N_{13})^2$;
3. for $A^1A^4B^2 + A^1A^4B^3$: $(N_{142}-1)(N_{143}-1)(N_{14})^2$;
4. for $A^2A^3B^1 + A^2A^3B^4$: $(N_{231}-1)(N_{234}-1)(N_{23})^2$;
5. for $A^2A^4B^1 + A^2A^4B^3$: $(N_{241}-1)(N_{243}-1)(N_{24})^2$;
6. for $A^3A^4B^1 + A^3A^4B^2$: $(N_{341}-1)(N_{342}-1)(N_{34})^2$;
7. for $A^2A^3B^1 + A^2A^4B^1$: $(N_{231}-1)(N_{241}-1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2$;

8. for $A^2A^3B^1 + A^3A^4B^1$: $(N_{231}-1)(N_{341}-1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2$;
9. for $A^2A^4B^1 + A^3A^4B^1$: $(N_{241}-1)(N_{341}-1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2$;
10. for $A^1A^3B^2 + A^1A^4B^2$: $(N_{132}-1)(N_{142}-1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2$;
11. for $A^1A^3B^2 + A^3A^4B^2$: $(N_{132}-1)(N_{342}-1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2$;
12. for $A^1A^4B^2 + A^3A^4B^2$: $(N_{142}-1)(N_{342}-1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2$;
13. for $A^1A^4B^3 + A^2A^4B^3$: $(N_{143}-1)(N_{243}-1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2$;
14. for $A^1A^4B^3 + A^1A^2B^3$: $(N_{143}-1)(N_{123}-1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2$;
15. for $A^2A^4B^3 + A^1A^2B^3$: $(N_{243}-1)(N_{123}-1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2$;
16. for $A^1A^2B^4 + A^1A^3B^4$: $(N_{124}-1)(N_{134}-1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2$;
17. for $A^1A^2B^4 + A^2A^3B^4$: $(N_{124}-1)(N_{234}-1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2$;
18. for $A^2A^3B^4 + A^1A^3B^4$: $(N_{234}-1)(N_{134}-1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2$.

For example, $A^1A^2B^3 + A^1A^3B^2$ is not allowed since B^3 and A^3 appear simultaneously. $A^1A^2B^3 + A^1A^2B^4$ is allowed which have $(N_{123}-1)(N_{124}-1)$ choices of nonzero coefficients. For each of them, by applying the exclusion principle (see the main text), one may attach $AAdA$ terms that do not contain both A^3 and A^4 , i.e., $A^1A^2dA^1$ and $A^1A^2dA^2$.

Next, if there are three nonzero AAB terms in the action, we have:

1. for $A^2A^3B^1 + A^2A^4B^1 + A^3A^4B^1$: $(N_{231}-1)(N_{241}-1)(N_{341}-1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2$;
2. for $A^1A^3B^2 + A^1A^4B^2 + A^3A^4B^2$: $(N_{132}-1)(N_{142}-1)(N_{342}-1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2$;
3. for $A^1A^4B^3 + A^2A^4B^3 + A^1A^2B^3$: $(N_{143}-1)(N_{243}-1)(N_{123}-1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2$;
4. for $A^1A^2B^4 + A^1A^3B^4 + A^2A^3B^4$: $(N_{124}-1)(N_{134}-1)(N_{234}-1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2$.

One can check that the above results exhaust all gauge theories that contain the BR braiding. Summing up the

above possibilities, we get the total number of gauge theories with BR braiding:

$$\begin{aligned}
& (N_{231}N_{241}N_{341} - 1)(N_{23})^2(N_{24})^2(N_{34})^2(N_{234})^2 \\
& + (N_{132}N_{142}N_{342} - 1)(N_{13})^2(N_{14})^2(N_{34})^2(N_{134})^2 \\
& + (N_{123}N_{143}N_{243} - 1)(N_{12})^2(N_{14})^2(N_{24})^2(N_{124})^2 \\
& + (N_{124}N_{134}N_{234} - 1)(N_{12})^2(N_{13})^2(N_{23})^2(N_{123})^2 \\
& + (N_{123} - 1)(N_{124} - 1)(N_{12})^2 \\
& + (N_{132} - 1)(N_{134} - 1)(N_{13})^2 \\
& + (N_{142} - 1)(N_{143} - 1)(N_{14})^2 \\
& + (N_{231} - 1)(N_{234} - 1)(N_{23})^2 \\
& + (N_{241} - 1)(N_{243} - 1)(N_{24})^2 \\
& + (N_{341} - 1)(N_{342} - 1)(N_{34})^2.
\end{aligned} \tag{31}$$

Section-4 Gauge transformations in the “almost trivial” field theory “ $bda + aab$ ” of SPTs

The gauge transformations for the action S_{aab} in Eq. (10) can be obtained by formally setting $N_1 = N_2 = N_3 = 1$ in Eq. (4) and (5):

$$a^i \rightarrow a^i + d\tilde{\chi}^i, a^3 \rightarrow a^3 + d\tilde{\chi}^3 + \tilde{\mathcal{Y}}_3, \tag{32}$$

$$b^3 \rightarrow b^3 + d\tilde{V}^3, b^i \rightarrow b^i + d\tilde{V}^i + \tilde{\mathcal{Y}}_i, \tag{33}$$

where $i = 1, 2$ only. Adding the tilde “ \sim ” is to differ the present gauge transformations from Eq. (4) and (5). Again, $\tilde{\chi}^{1,2,3}$ and $\tilde{V}^{1,2,3}$ satisfy the usual “winding number” conditions:

$$\int_{\mathbb{S}^1} d\tilde{\chi}^{1,2,3} \in 2\pi\mathbb{Z}, \tag{34}$$

$$\int_{\mathbb{S}^2} d\tilde{V}^{1,2,3} \in 2\pi\mathbb{Z}. \tag{35}$$

$\tilde{\mathcal{Y}}_{1,2,3}$ are *nontrivially* induced by the $a^1 a^2 b^3$ term ($i, j = 1, 2$ only):

$$\tilde{\mathcal{Y}}_i = \frac{c}{2\pi} \epsilon^{ij} (a^j \tilde{V}^3 - \tilde{\chi}^j b^3 - \tilde{\chi}^j d\tilde{V}^3), \tag{36}$$

$$\tilde{\mathcal{Y}}_3 = -\frac{c}{2\pi} (\epsilon^{ij} \tilde{\chi}^i a^j + \tilde{\chi}^1 d\tilde{\chi}^2). \tag{37}$$

Under the above gauge transformations, S_{aab} is gauge invariant. However, we need to investigate whether or not S_{sym} in Eq. (11) is gauge invariant. After the gauge transformations ($A^{1,2}$ and B^3 transform under Eq. (4) and (5), S_{sym} is changed by:

$$\Delta S_{\text{sym}} = - \int A^1 d\tilde{\mathcal{Y}}_1 - \int A^2 d\tilde{\mathcal{Y}}_2 - \int B^3 d\tilde{\mathcal{Y}}_3. \tag{38}$$

In order to guarantee that $\Delta S_{\text{sym}} = 0 \bmod 2\pi$, one can check that the quantization requirement for c must be enlarged from $c = \mathbb{Z}$ (in absence of S_{sym}) to $c = \frac{N_1 N_2 N_3}{N_{123}} \mathbb{Z}$

(in the presence of S_{sym}) under Eqs.(34,35) and the following conditions:

$$\int_{\mathbb{S}^1} A^1 = \frac{2\pi}{N_1} \mathbb{Z}, \int_{\mathbb{S}^1} A^2 = \frac{2\pi}{N_2} \mathbb{Z}, \tag{39}$$

$$\int_{\mathbb{S}^2} B^3 = \frac{2\pi}{N_3} \mathbb{Z}, \tag{40}$$

$$\int_{\mathbb{S}^1} a^1 = 2\pi\mathbb{Z}, \int_{\mathbb{S}^1} a^2 = 2\pi\mathbb{Z}, \tag{41}$$

$$\int_{\mathbb{S}^2} b^3 = 2\pi\mathbb{Z}, \tag{42}$$

$$da^1 = 0, da^2 = 0, db^3 = 0. \tag{43}$$

Conditions (39, 40) are the definitions of the probe fields A^1, A^2, B^3 (see the main text around S_{sym}). Conditions (41, 42, 43) can be obtained by studying the equations of motion of b^1, b^2, a^3 . Among them, conditions (43) mean that all gauge fields in the ground states of discrete gauge theories are flat.

From the above derivation, we see that, although c in S_{aab} before imposing symmetry can be arbitrary integral, only subset of \mathbb{Z} , i.e. $c = \frac{N_1 N_2 N_3}{N_{123}} \mathbb{Z}$ is allowed after imposing symmetry. This line of thinking was previously explained in details in SPT field theory description *within* cohomology [15].

Section-5 Coexistence of loop condensations and boson condensations

SPTs can be understood as a result of symmetry domain wall proliferations in the symmetry-breaking condensations. Previous studies mainly focus on SPTs with usual 0-form symmetry. Below, for readers who are interested in this line of thinking, we sketch the key steps toward the condensation picture of SPTs protected by mixed global symmetry. The symmetry group is $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ where $\mathbb{Z}_{N_1}, \mathbb{Z}_{N_2}$ are 0-form symmetries and \mathbb{Z}_{N_3} is a 1-form symmetry. Consider a state with broken $U^3(1)$ symmetry in which first two copies of $U(1)$ are 0-form symmetries and the last copy of $U(1)$ is a 1-form symmetry. It describes a mixture of two particle condensates and a loop condensate. We bind the line charge to the intersection line of the symmetry-breaking domain walls of the two 0-form symmetries. Let the compact phase fluctuation for the two particle condensates be the 0-forms θ^1 and θ^2 and that for the loop condensate be the 1-form Θ^3 . The system is essentially a $U^3(1)$ non-linear σ -model with a multi-kink topological term $\frac{c}{(2\pi)^2} d\theta^1 d\theta^2 d\Theta^3$ describing the charge binding. The action is written as ($i, j = 1, 2$; $g_{1,2,3}$ are coupling

constants)

$$S = \int \sum_{i=1}^2 \frac{g_i}{2} d\theta^i \star d\theta^i + \int \frac{g_3}{2} d\Theta^3 \star d\Theta^3 \\ + \int \frac{c}{(2\pi)^2} d\theta^1 d\theta^2 d\Theta^3 . \quad (44)$$

Then, we recover the symmetry by disordering the system. Due to the multi-kink term, the $U^3(1)$ global symmetry generally becomes anomalous [15, 43]. If $c = N_i N_j N_k t / N_{ijk}$ for some integers N_i, N_j, N_k and integer t defined mod N_{ijk} , the system possesses anomaly-free $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$ symmetry. We decompose the compact phase fluctuations θ^1, θ^2 and Θ^3 into smooth part and singular part, that is $d\theta^1 = d\theta_s^1 + a^1$, $d\theta^2 = d\theta_s^2 + a^2$ and $d\Theta^3 = d\Theta_s^3 + b^3$, where the 1-forms a^1, a^2 and 2-form b^3 are compact gauge fields describing the singular phase fluctuations. Then the action becomes

$$S = \int \sum_{i=1}^2 \int \frac{g_i}{2} d(\theta_s^i + a^i) \star d(\theta_s^i + a^i) \\ + \int \frac{g_3}{2} d(\Theta_s^3 + b^3) \star d(\Theta_s^3 + b^3) \\ + \int \frac{c}{(2\pi)^2} d(\theta_s^1 + a^2) d(\theta_s^1 + a^2) d(\Theta_s^3 + b^3) . \quad (45)$$

By using the standard particle-vortex duality technique ([15] for particle condensations; [42] for loop condensations), one may integrate out smooth parts of the phase fluctuations during which several new fields (two-form $b^{1,2}$ and 1-form a^3) are introduced to enforce current conservation

$$S = \int \sum_{i=1}^3 \frac{1}{2\pi} b^i da^i + \int \frac{c}{(2\pi)^2} a^1 a^2 b^3 , \quad (46)$$

where the 2-forms b^1, b^2 and 1-form a^3 are non-compact gauge fields appear under duality. Physically, $\frac{1}{2\pi} \star db^1$, $\frac{1}{2\pi} \star db^2$ and $\frac{1}{2\pi} \star da^3$ are the charge current for the \mathbb{Z}_{N_1} , \mathbb{Z}_{N_2} 0-form symmetries and \mathbb{Z}_{N_3} 1-form symmetry respectively.